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Short Communication

Nonlinear free vibration analysis of square plates with various boundary conditions

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Abstract

Non-linearity in plate vibration problems may arise out of material and geometric non-linearity. The present study focuses only on geometric non-linearity. A new methodology is proposed that can be employed for plate structure problems having any combination of boundary conditions to determine the non-linear frequencies and mode shapes. Large amplitude vibration problem is analysed in two parts. The static problem corresponding to a uniform transverse loading is solved first and the dynamic problem is subsequently taken up with the known deflection field. Both these problems are formulated through energy method, the underlying principle being the extremisation of total energy of the system in its equilibrium state. The solution methodology employs an iterative numerical scheme using the technique of successive relaxation. The results of non-linear analyses are validated with the published results and excellent agreement is observed. The solution methodology can be applied to any kind of boundary condition as pointed out and the back-bone curves documented may be used by the practicing engineers as design curves.

1. Introduction

Nonlinearity in plate vibration problems may arise out of both material nonlinearity and geometric nonlinearity. The first one is due to the nonlinear stress–strain behaviour of material,

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whereas large deflections give rise to geometric nonlinearity. Solution of such problems, which are much more complicated compared to their linear counterpart, was first obtained analytically by Von Karman [1] for simply supported boundary condition. The method was extended by Chu and Herrmann [2] and they obtained the in-plane displacement fields by employing double Fourier series. For solution of generalised problem, other methods have been proposed by various researchers. Leissa [3] has mentioned some of them; a comprehensive review is carried out by Sathyamoorthy [4].

Among the recent research work on large amplitude vibration, Kobayashi and Leissa [5] observed the behaviour of a rectangular shallow shell supported on shear diaphragms by using Galerkin principle. Wang et al. [6] used a numerical technique based on boundary element method to solve the static large deflection problems of thin elastic plates. Benamar et al. [7,8] examined, theoretically and experimentally, the dynamic behaviour of fully clamped rectangular plates under large amplitudes of vibration. Han and Petyt [9] studied nonlinear vibration frequency and mode shape of thin isotropic and laminated plates. Later Ribeiro and Petyt [10] have analysed the problem with internal resonance. Elbeyli and Anlas [11] carried out an analytical study to determine the nonlinear response of a simply supported plate under transverse harmonic excitation by using the method of multiple scales. Raju et al. [12] carried out experimental and theoretical investigations on large amplitude free vibration analysis of square clamped plate subjected to transverse loading.

Review of the literature indicates that different researchers studied the nonlinear vibration problem with some specific boundary conditions. Quite a few different techniques have been used to obtain the natural frequencies and mode shapes. Under this context, the need is felt for development of a general-purpose method to identify the nonlinear frequencies and mode shapes of plate for any kind of boundary condition. The present authors [13] developed a new methodology that can be employed to plate structure having any kind of boundary conditions and their combinations as well.

The necessary higher-order constitutive functions are formed by following Gram–Schmidt orthogonalisation procedure, thus making the solution space complete. The solution of the dynamic problem is obtained through the solution of static deflection field. The static deflection field required in the present method is obtained for the plate considering uniform transverse loading condition. The nonlinear static problem is solved by employing an iterative method with an appropriate relaxation technique.

The vibration mode shapes for all the cases are presented corresponding to the minimum and maximum amplitudes of vibration. It is observed that the present methodology is quite robust, stable and realistic and the backbone curves documented may be used by practicing engineers as design curves.

2. Analysis

In the present paper, the large amplitude vibration problem is analysed in two parts. The static problem corresponding to a uniform transverse loading is solved first and the dynamic problem is subsequently taken up with the known deflection field. Both the static and dynamic problems are formulated through energy method, the underlying principle being the extremisation of total energy of the system in its equilibrium state. The plate analysed is assumed to be linearly elastic, homogeneous, of uniform thickness and sufficiently thin such that the effect of transverse shear deformation can be neglected. The formulation is carried out for a rectangular plate, but the results are furnished for square plate, in particular.

A rectangular plate ($a \times b \times h$) when subjected to a transverse load produces deflection and the energies stored in the plate are the strain energy due to bending (U_b) and the membrane strain energy due to stretching (U_m). Thus the total strain energy in the plate is $U = U_b + U_m$. Expressions for U_b and U_m are given in Eqs. (1) and (2), respectively.

$$U_b = \frac{D}{2a^2\lambda}\int_0^1\int_0^1 \{(w_{,\xi\xi} + \lambda^2 w_{,\eta\eta})^2 + 2\lambda^2(1-\upsilon)(w^2_{,\xi\eta} - w_{,\xi\xi}w_{,\eta\eta})\}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{1}$$

$$\begin{aligned} U_m = \frac{Eh}{2a^2\lambda(1-\upsilon)}\int_0^1\int_0^1 \Big\{ & [a^2u^2_{,\xi} + au_{,\xi}w^2_{,\xi} + a^2\lambda^2 v^2_{,\eta} + a\lambda^3 v_{,\eta}w^2_{,\eta}] + \frac{1}{4}[w^2_{,\xi} + \lambda^2 w^2_{,\eta}]^2 \\ & + 2\upsilon\left[a^2\lambda u_{,\xi}v_{,\eta} + \frac{1}{2}a\lambda^3 v_{,\eta}w^2_{,\eta} + \frac{1}{2}au_{,\xi}w^2_{,\xi}\right] \\ & + \frac{1-\upsilon}{2}[a^2\lambda^2u^2_{,\eta} + a^2\lambda u_{,\eta}v_{,\xi} + a^2v^2_{,\xi} + 2a\lambda^2u_{,\eta}w_{,\xi}w_{,\eta} + 2a\lambda^2 v_{,\xi}w_{,\xi}w_{,\eta}]\Big\}\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{2}$$

where E, υ, ρ and $D(= Eh^3/12(1-\upsilon^2))$ are elastic modulus, Poisson's ratio, density and flexural rigidity of the plate, respectively. The notation $\lambda(= a/b)$ in the equations is the plate aspect ratio. The total potential energy (V) due to external transverse loading is given by

$$V = \frac{a^2}{\lambda}\int_0^1\int_0^1 -(qw)\,\mathrm{d}\xi\,\mathrm{d}\eta - \sum P_i w_i, \tag{3}$$

where q is distributed load and P_i is the ith concentrated load at (ξ_i, η_i).

In Eqs. (1)–(3), the mid-plane coordinates are expressed in dimensionless form as $\xi = x/a$, $\eta = y/b$, to facilitate the computation work, while the dimensions of all other physical quantities, like load, deflection, etc. are retained as such.

3. The static problem

From the principle of conservation of total energy of the system $\delta(U + V) = 0$, the governing differential equation for the static problem is obtained. The displacement fields of the plate, u, υ and w are expressed by linear combinations of unknown parameters d_i as follows:

$$w(\xi,\eta) = \sum_{i=1}^{nw} d_i\phi_i\psi_i, \quad u(\xi,\eta) = \sum_{i=1+nw}^{nw+nu} d_i\alpha_i\psi_i \quad \text{and} \quad \upsilon(\xi,\eta) = \sum_{i=1+nw+nu}^{nw+nu+nv} d_i\phi_i\beta_i,$$

where $\phi(\xi)$, $\psi(\eta)$, $\alpha(\xi)$ and $\beta(\eta)$ are sets of orthogonal functions. The starting functions of these orthogonal sets satisfy the corresponding boundary conditions of the plate system.

3.1. Generation of starting functions

Beam deflection functions are derived from static deflection shape of the plate under uniform loading.These are the starting function of the orthogonal sets for $\phi(\xi)$ and $\psi(\eta)$ corresponding to the boundary condition of the plate along the particular coordinate axis. It is found that six basic beam functions are necessary for all the boundary conditions referred in Ref. [14] and both transcendental as well as polynomial functions are used for the descriptions of such functions [15]. The higher-order functions are generated through a numerical implementation of Gram–Schmidt orthogonalisation procedure and thus they take care of the higher-order vibration modes. To cater to the need of the numerical scheme, all the functions are described numerically at some suitably selected Gauss points. The details of the procedure of the generation of the function are presented in Ref. [13]. For S-F beam, the necessary rigid body modes are incorporated by adding appropriate functions to the corresponding sets of orthogonal functions. The starting functions for $\alpha(\xi)$ and $\beta(\eta)$ are obtained from zero displacement in-plane boundary conditions ($u = 0$ at $\xi = 0$ and 1 and $v = 0$ at $\eta = 0$ and 1) and hence they are assumed as $\xi(\xi - 1)$ and $\eta(\eta - 1)$, respectively.

3.2. Solution methodology

In the present paper, results are generated for a plate under uniform loading. Therefore, in the solution methodology, the terms for concentrated loads (P_i) are dropped. However, they have been considered for the purpose of comparison only. Substituting the assumed series solutions for u, v and w in the expressions of U and V, one can obtain the system governing equations in matrix form $[K]\{d\} = q\{R\}$. The unknown coefficients can be obtained from $\{d\} = q[K]^{-1}\{R\}$ through an iterative scheme [13] using a relaxation technique. In each step of the iteration, the error vector $\{\varepsilon\} = q[K]^{-1}\{R\} - \{d\}$ is computed. If error is not within the permitted value of tolerance, the process is repeated with new values of $\{d\}$ until $\{\varepsilon\}$ becomes less than the specified tolerance.

4. The dynamic problem

The dynamic problem is formulated following Hamilton's principle, $\delta \int (T - U)\,\mathrm{d}t = 0$, where the strain energy U corresponds to the deflected shape of the plate and the kinetic energy (T) is expressed as

$$T = \frac{1}{2}\rho h \frac{a^2}{\lambda} \int_0^1 \int_0^1 (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4}$$

The dynamic displacements $w(\xi, \eta, t)$, $u(\xi, \eta, t)$ and $v(\xi, \eta, t)$ are assumed to be separable in space and time. They are constituted through a new set of unknown parameters $\{d\}$ as $\sum_{i=1}^{nw} d_i \phi_i \psi_i \gamma_i$, $\sum_{i=1}^{nu} d_{i+nw} \alpha_i \psi_i \gamma_{i+nw}$ and $\sum_{i=1}^{nv} d_{i+nw+nu} \phi_i \beta_i \gamma_{i+nw+nu}$, respectively. The space functions are completely known from the static analysis and the set of temporal function is expressed by $\gamma_i(t) = \mathrm{e}^{\mathrm{i}\varpi t}$. Substituting the above series in Eq. (4), the governing equation of the dynamic system can be written in the form

$$-\omega^2[M]\{d\} + [K]\{d\} = 0. \tag{5}$$

$[K]$ and $[M]$ are symmetric square matrices of order $(nw + nu + nv)$, the details of which are given in Ref. [13]. The standard eigenvalue problem of Eq. (5) is solved numerically for the natural frequencies ω_i by using IMSL routines.

5. Results and discussions

A large amplitude free vibration study is carried out for square plates under uniform pressure loading. The solution methodology employs an iterative numerical scheme using the technique of successive relaxation. It is observed that the convergence of the numerical iteration scheme depends on the tolerance value of the error limit, ε, and is presented in Ref. [13]. The results pertaining to the combinations of clamped (C) and free (F) boundary conditions are also presented there. In this paper, the effect of different boundary conditions that may arise from different combinations of clamped (C), simply supported (S) and free (F) edges of a plate are studied. The boundary condition of the plate is specified, through its edge conditions (C, S or F), in anticlockwise direction starting from the edge $x = 0$. The numerical study is carried out for a 400 mm square steel plate of 2.5 mm thickness having $E = 2.1 \times 10^{11}$ Pa, $\rho = 7850\,\mathrm{Kg/m^3}$ and $v = 0.3$.

For the clamped boundary condition, the results of geometrically nonlinear vibration analysis are compared with those of different researchers and are presented in Table 1. In case of simply supported boundary conditions, variation of nonlinear frequency with amplitude of motion is compared to that of Chia [18]. However, he has furnished the plot for the first nonlinear frequency only. The result of Chia is obtained by digitizing the relevant graph and the discrete values are indicated as data points in Fig. 1(a). It may be noted that the present results match quite well to that of the other researchers both for clamped and simply supported boundary conditions.

Figs. 1(a)–(f) show the nonlinear frequency amplitude relationships (backbone curves) of the plate having typical combinations of boundary conditions. Each of these figures represents first six vibration modes in dimensionless form. The ratio of the maximum plate deflection to plate thickness is taken as the dimensionless amplitude $w^*(= W_{\max}/h)$ while the nonlinear frequency is normalized (ω_{nl}/ω_1) by the corresponding fundamental linear frequency (ω_1). However, SFFF boundary conditions give rise to rigid body modes of vibration [19], the corresponding frequency parameters being zero. Hence, SFFF plate is normalised with respect to ω_2.

It is observed that with decreasing rigidity of the plate, as the number of free edges in the boundary condition increase, the backbone curves for different modes spread apart. In an earlier

Table 1
Comparison of nonlinear frequency ratios of a square plate with all edges clamped

	$W_{\max}/h$				
	0.2	0.4	0.6	0.8	1.0
Lau et al. [16]	1.0196	1.0763	1.1645	1.2779	1.4109
Chandrasekharappa and Srirangarayan [17]	1.0143	1.0572	1.1288	1.2290	1.3578
Present study	1.026	1.098	1.233	1.381	1.568

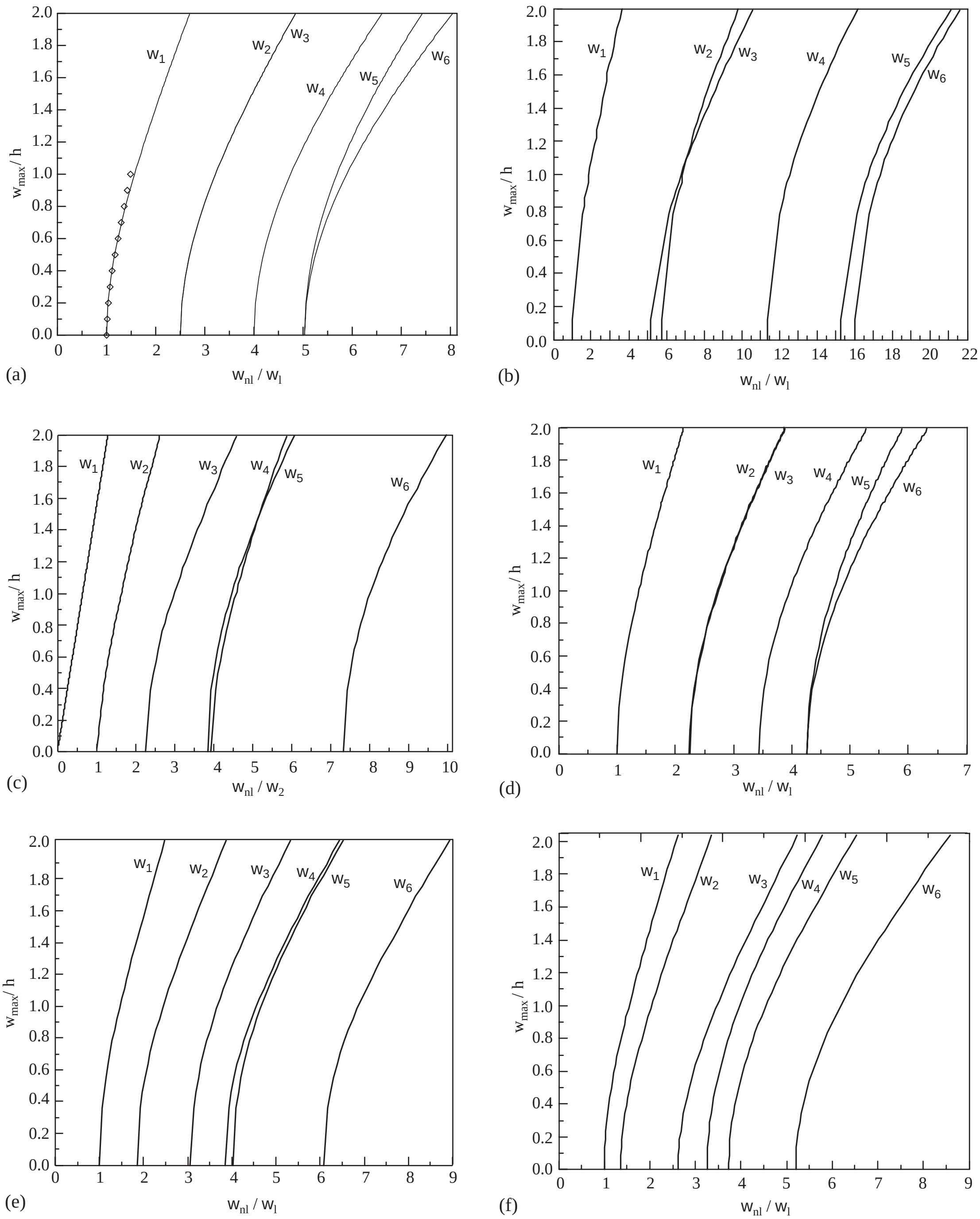


Fig. 1. Backbone curves for (a) SSSS boundary conditions (◇: points taken from Ref. [18]); (b) SSFF boundary conditions; (c) SFFF boundary conditions; (d) CCSS boundary conditions; (e) CSSF boundary conditions; (f) CFSF boundary conditions.

study [20], it is observed that the degree of nonlinearity has a pronounced effect on the frequency parameter. Thus it can be stated that plates with different boundary conditions possess different degrees of nonlinearity.

The effect of vibration amplitude on the dynamic behaviour of the plate is highlighted also in the mode shape plots for all the six vibration modes. Both linear (corresponding to $w^* = 0$) and nonlinear (corresponding to $w^* = 2.0$) mode shapes for two specific boundary conditions (SFFF and CCSS) are shown in Figs. 2(a) and (b). The degree of nonlinearity manifested through a change in a particular mode shape, appears to be more pronounced for higher modes of vibration.

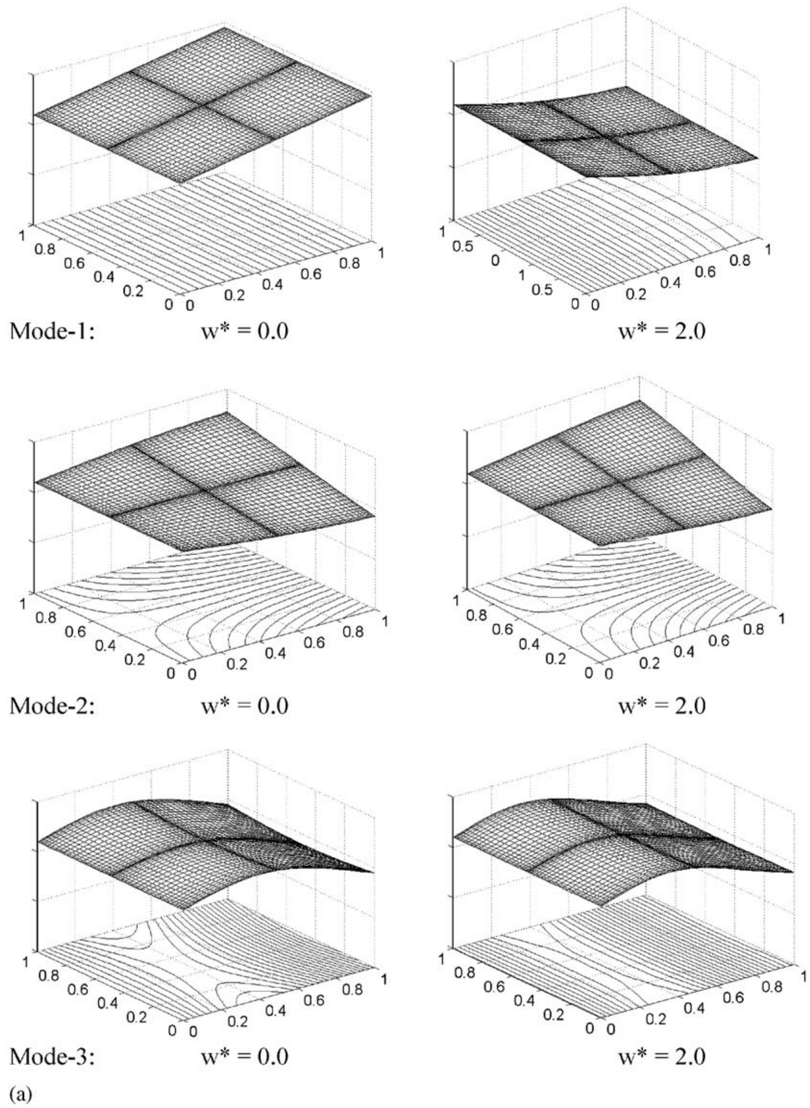


Fig. 2. Mode shape plots for (a) SFFF boundary condition; (b) CCSS boundary condition.

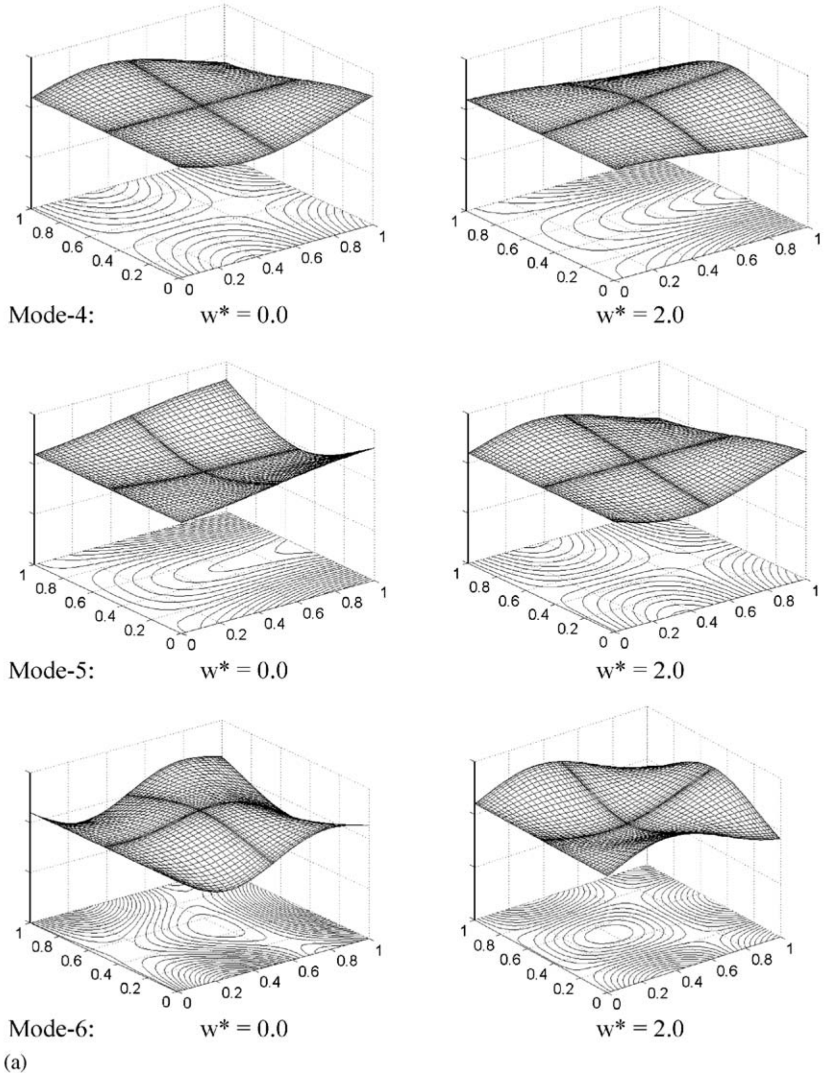
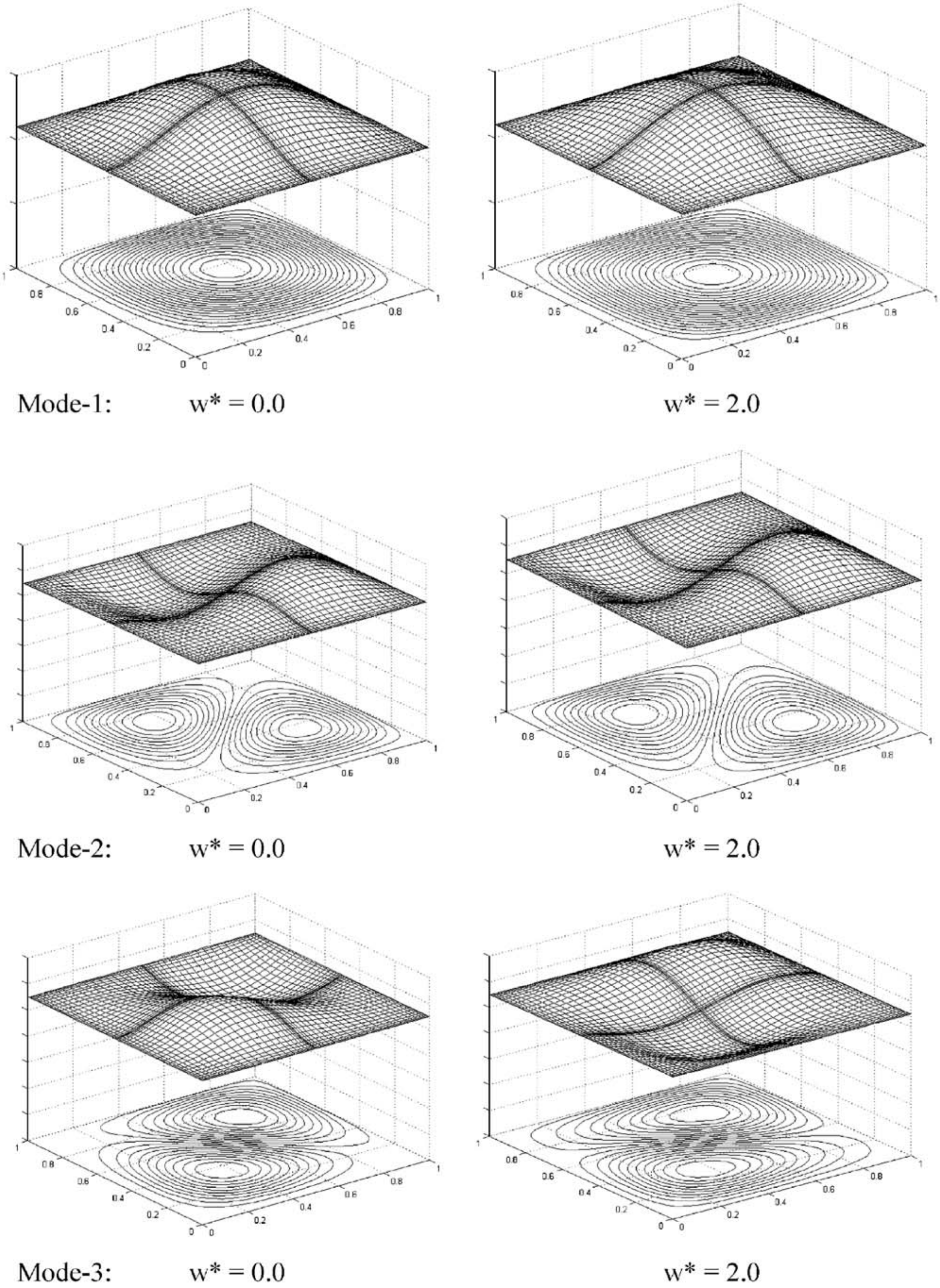


Fig. 2. (*Continued*)

Phenomenon of mode switching with amplitude of vibration is observed for some specific boundary conditions only, as shown in Figs. 1(a)–(d). This phenomenon has been appropriately supported in the corresponding mode shape plots (Figs. 2(a) and (b)). A similar trend of mode switching phenomenon has been observed by Singh [20] for a trapezoidal shallow shell with CFCF boundary condition.

In Figs. 2(a) and (b), quite a few mode shape plots are apparently found to be identical both for linear ($w^* = 0.0$) and nonlinear ($w^* = 2.0$) cases. However, a micro-level study is also carried out for both the boundary conditions and the corresponding mode shape plots, taken at a suitable cross-section of the vibrating plate, and are presented in Figs. 3(a) and (b), respectively. In the



(b)

Fig. 2. (*Continued*)

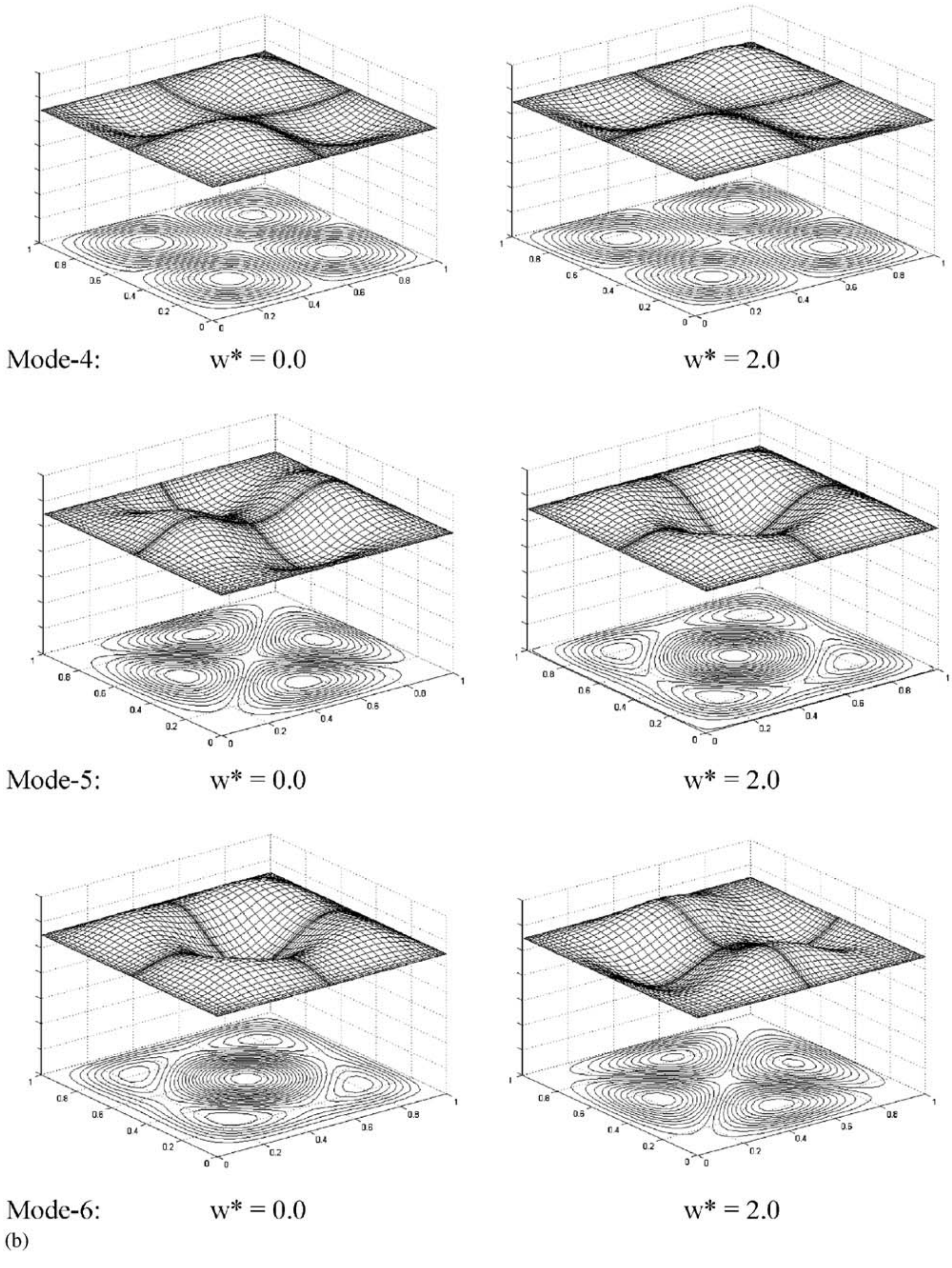


Fig. 2. (*Continued*)

study, further elaboration is obtained by including the mode shape corresponding to $w^* = 1.0$. The figures reveal that effect of nonlinearity on mode shape is quite appreciable as w^* is increased from 0 to 2.0. In fact this effect is more pronounced as the boundary conditions are changed from rigid to flexible.

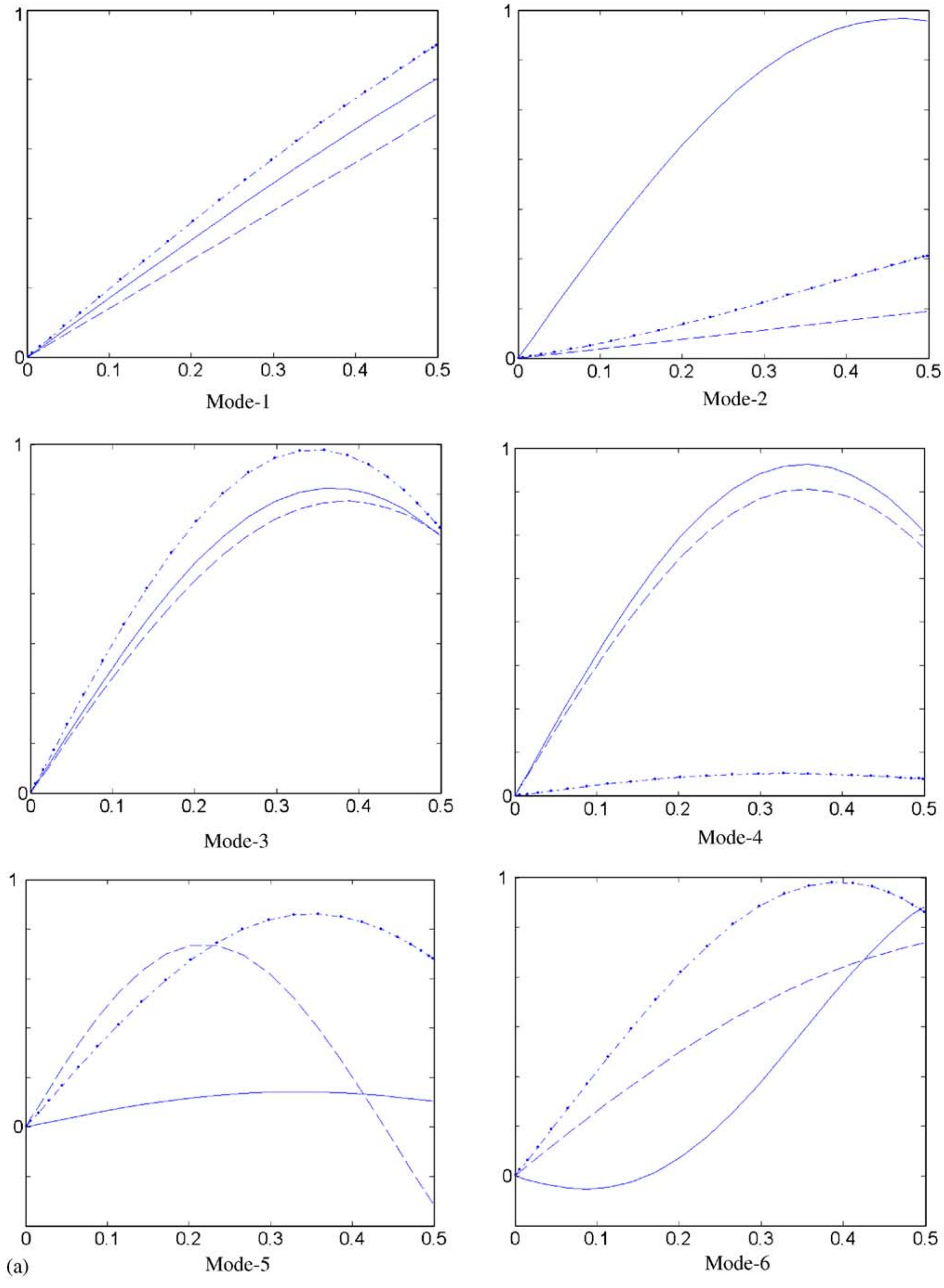


Fig. 3. Effect of amplitude on mode shape taken at a suitable cross section of a vibrating plate with (a) SFFF boundary condition; _ _ _ _ $W^* = 0.0$, ______ $W^* = 1.0$, _ . _ . _ $W^* = 2.0$. (b) CCSS boundary condition; _ _ _ _ $W^* = 0.0$, ______ $W^* = 1.0$, _ . _ . _ $W^* = 2.0$.

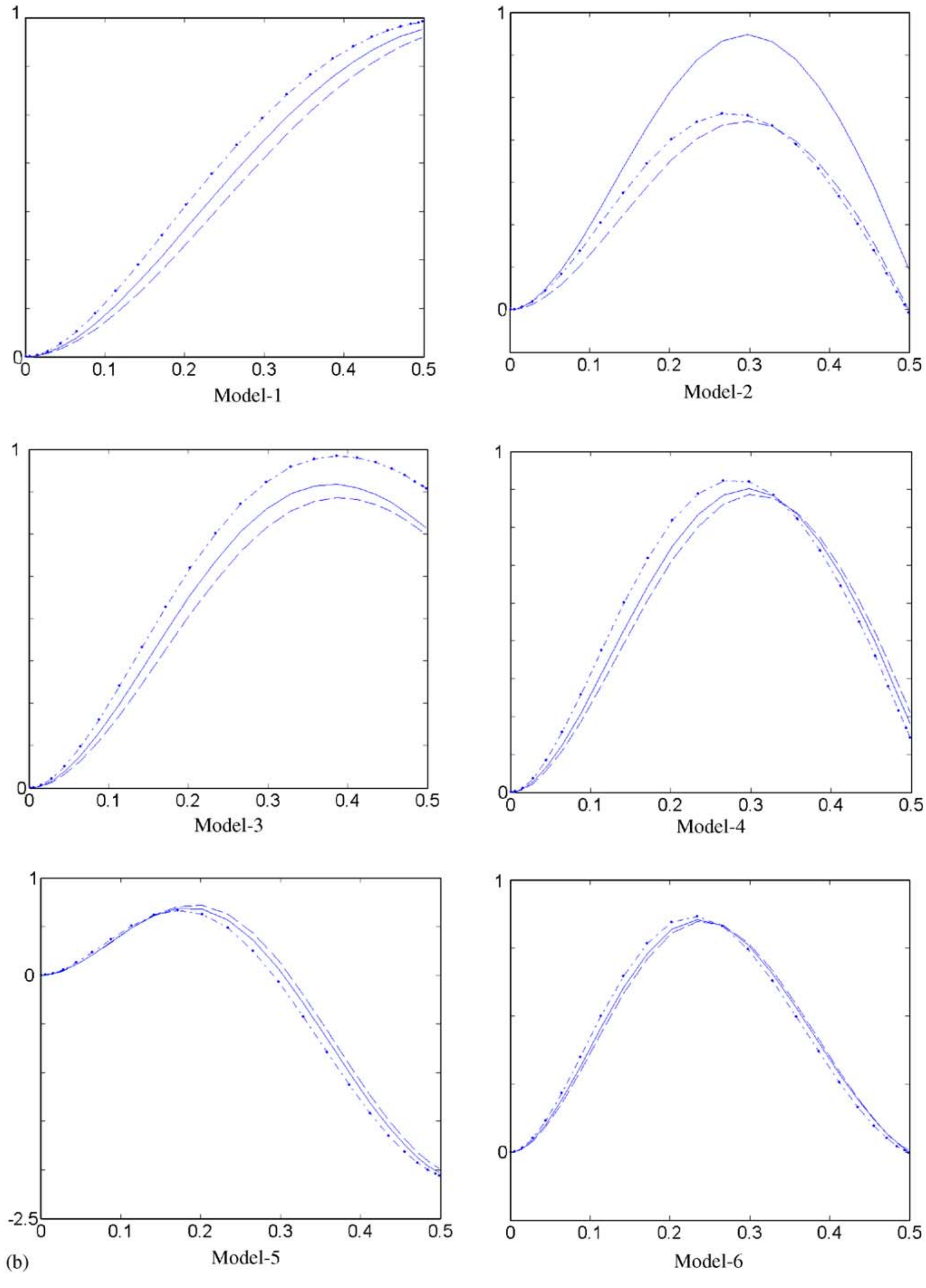


Fig. 3. (*Continued*)

6. Conclusion

Large amplitude free vibration analysis of a thin square plate with different nonclassical boundary conditions is investigated. The study is carried out following a novel method in which static analysis serves as the basis for the subsequent dynamic study. The results of nonlinear analyses are validated with the published results of other researchers and excellent agreement is observed. It can be applied to any kind of boundary condition as pointed out in the present paper and the backbone curves documented may be used by the practicing engineers as design curves.

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